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Some minimax problems for graphs

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Abstract

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If a characteristic of a simple graph G allows an extension to nonnegative edge valuations of G, the corresponding absolute characteristic is defined as the extreme of the characteristic over all nonnegative edge valuations of G with an average value of 1. A survey of the results for previously studied cases is given and new results on the absolute algebraic connectivity, absolute diameter and absolute radius of a tree are added.

1. Introduction

In [2–5], a program was started to find new characteristics of a finite nondirected graph $G=(V,E)$ by using the following idea.

Suppose a characteristic $\omega(G)$ of G can be generalized to $\omega(G_C)$ for the class of edge-valuated graphs $G_C=(V,E,C)$, where C is a valuation of the set of edges E by nonnegative numbers with the sum $|E|$. (We denote this set of valuations by $\mathscr{C}(G)$.) Then we obtain a characteristic $\hat{\omega}(G)$ by minimizing or maximizing $\omega(G_C)$ over all $C\in\mathscr{C}(G)$.

For example, if $\omega(G)$ means the maximum degree $d(G)$ of a vertex in G, then

$$d(G_C)=\max_{i\in V}\sum_{(i,k)\in E} c_{ik}, \tag{1}$$

where c_{ik} is the value assigned to the edge (i, k), and

$$\hat{d}(G)=\min_{C\in\mathscr{C}(G)} d(G_C).$$

To survey some of the results, denote further by $\delta(G)$ the minimum degree of the vertices of G (and $\hat{\delta}(G)=\max_{C\in\mathscr{C}(G)}\delta(G_C)$, where $\delta(G_C)$ is defined analogously to (1)),

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by $\gamma(G)$ the edge connectivity of G; for $\gamma(G_C)$,

$$\gamma(G_C) = \min_{\emptyset \neq W \subsetneq V} \sum_{\substack{i \in W \\ k \notin W}} c_{ik}$$

and

$$\hat{\gamma}(G) = \max_{C \in \mathscr{C}(G)} \gamma(G_C).$$

Similarly, the maximum cut in G_C is

$$c(G_C) = \max_{W \subset V} \sum_{\substack{i \in W \\ k \notin W}} c_{ik},$$

$$\hat{c}(G) = \min_{C \in \mathscr{C}(G)} c(G_C).$$

Further characteristics correspond to the adjacency matrix $A(G)$ and to the Laplacean matrix $L(G)$. For G_C,

$$A(G_C) = (c_{ik}),$$

where c_{ik} is set to zero if $(i, k) \notin E$ (also $c_{ii} = 0$) and c_{ik} is the value assigned to the edge (i, k) if $(i, k) \in E$.

Denoting by $\Delta(G_C)$ the diagonal matrix the ith diagonal entry of which is the degree of the ith vertex, i.e. the sum of the values of all edges incident to this vertex, the Laplacian matrix $L(G_C)$ is then

$$L(G_C) = \Delta(G_C) - A(G_C).$$

Thus, it is the matrix of the positive-semidefinite quadratic form

$$\sum_{\substack{(i,k) \in E \\ i < k}} c_{ik}(x_i - x_k)^2.$$

It was shown in [1] that the second smallest eigenvalue $a(G_C)$ (the smallest is zero and the corresponding eigenvector has all coordinates equal), called algebraic connectivity, satisfies the inequalities

$$2\left(1 - \cos\frac{\pi}{n}\right)\gamma(G_C) \leqslant a(G_C) \leqslant \frac{n}{n-1}\gamma(G_C).$$

Therefore, by maximizing over all $C \in \mathscr{C}(G)$,

$$2\left(1 - \cos\frac{\pi}{n}\right)\hat{\gamma}(G) \leqslant \hat{a}(G) \leqslant \frac{n}{n-1}\hat{\gamma}(G), \tag{2}$$

and these inequalities are the best possible in the class of all connected graphs with n vertices.

In [4], the maximum eigenvalue $b(G_C)$ of the Laplacian matrix and its minimum $\hat{b}(G_C) = \min_{C \in \mathscr{C}(G)} b(G_C)$, in [5] the spectral radius $\rho(G_C)$ and its maximum $\hat{\rho}(G)$ have been studied.

Let us just recall mutual inequalities between the mentioned characteristics (the number m means the number of edges $|E|$) of G:

$$\hat{\gamma}(G) \leqslant \hat{\delta}(G) \leqslant \frac{2m}{n} \leqslant \hat{\rho}(G) \leqslant \hat{d}(G) \leqslant \hat{c}(G) \leqslant \frac{n}{4}\hat{b}(G) \leqslant \frac{n}{2}\hat{a}(G), \tag{3}$$

which, together with

$$\hat{d}(G) \leqslant \hat{b}(G) \tag{4}$$

and (2), yields the best possible inequalities in the class of graphs with n vertices and m edges. (The inequality between $\hat{c}(G)$ and $\hat{b}(G)$ comes from a result by Mohar and Poljak [7].)

2. More about absolute algebraic connectivity of trees

In this section, we shall strengthen some results from [3].

First of all, a connected graph $G=(V,E)$ is, analogously to [6], considered as a metric space G_m whose points consist of all vertices of G as well as all points on the edges; such a point X on the edge (i, k) will be determined by its distances x_1, x_2 from i and k, which satisfy $x_1>0$, $x_2>0$, $x_1+x_2=1$. The distance of any two vertices p, q from V is defined as usual as the number of edges in the shortest path from p to q. One sees easily that one can extend the definition of the distance $d(P, Q)$ of any two points P, Q in G_m as the length of the shortest path between P and Q.

The absolute centre of gravity of G_m is then every point M in G_m for which the function

$$S(X)=\sum_{k\in V} d^2(X, k)$$

attains its minimum. Let us denote by $\sum(G)$ this minimum.

It was proved in [3] that for a tree T, $\hat{a}(T)$ can be expressed by

$$\hat{a}(T)=\frac{n-1}{\sum(T)}, \tag{5}$$

and, in addition, the centre of gravity M in T_m is uniquely determined. The following alternative holds:

(i) M is a vertex, $M\in V$, for which

$$S(M)\leqslant S(k)-n$$

for all vertices $k\neq M$, $k\in V$, holds.

(ii) M is point of the edge $(p, q)\in E$ for which

$$|S(p)-S(q)|<n \tag{6}$$

holds, and then

$$d(M,p)=\frac{1}{2n}(n+S(p)-S(q)),$$

$$d(M,q)=\frac{1}{2n}(n+S(q)-S(p)),$$

$$S(M)=\frac{1}{4n}(4n\,S(q)-(n+S(q)-S(p))^2). \tag{7}$$

(In fact $S(p)$ and $S(q)$ are in this case the two smallest numbers among $S(k)$, $k\in V$, and they differ by less than n.)

As a corollary, it was remarked in [3] that $\hat{a}(T)$ is for any tree a rational number.

We intend to strengthen this result. First, we shall prove: the following theorem:

Theorem 2.1. *Let T be a tree with n vertices, $n\geqslant 3$. Then*

$$\hat{a}(P_n)\leqslant\hat{a}(T)\leqslant\hat{a}(S_n).$$

In other words,

$$\frac{12}{n(n+1)}\leqslant\hat{a}(T)\leqslant 1.$$

Proof. For $n=3$, the result is true. Let, therefore, $n\geqslant 4$. By formula (5), it suffices to prove the inequalities

$$n-1\leqslant\textstyle\sum(T)\leqslant\frac{1}{12}n(n^2-1). \tag{8}$$

The left inequality in (8) is true if the absolute centre of gravity M of T is in a vertex. Suppose now that M is an interior point of an edge (i,k); let p be the number of vertices of T different from i from which the path to k passes through i, q the number of vertices different from k from which the path to i passes through k. Hence,

$$p+q=n-2. \tag{9}$$

Neither i nor k can be an end vertex of T by (6). Therefore,

$$p\geqslant 1,\quad q\geqslant 1. \tag{10}$$

By symmetry, we can suppose that the distance x from M to i is at most $\frac{1}{2}$.

Now,

$$\begin{aligned}S(M)&\geqslant x^2+(1-x)^2+p(1+x)^2+q(2-x)^2\\&\geqslant\tfrac{1}{2}+p+q(\tfrac{3}{2})^2,\end{aligned}$$

which is greater than $n-1$ by (9) and (10).

To prove the right inequality in (8), consider first the problem of minimizing the function

$$u(X)=\sum_{t\in V} d(X, t)$$

over all points X of the metric space T_m.

The following lemma is easy to prove.

Lemma 2.2. *Let (i, k) be an edge in T. Then*

$$u(i)<u(k)$$

if and only if the subset $V_{ik}\subset V$, consisting of exactly those vertices $p\in V$ for which the path from i to p contains k, has at most $n/2$ vertices. Also, $u(i)<u(k)$ if and only if $|V_{ik}|<n/2$.

Corollary 2.3. *The following alternative holds:*

(i) *the function $u(x)$ attains its minimum at a single point w. Then $w\in V$ and each of the, say, t components $C_1, \ldots, C_t$ $(t\geqslant 2)$ of the graph obtained from T by deleting the vertex w and all the incident edges, has at most $\frac{1}{2}(n-1)$* vertices;

(ii) *the function $u(x)$ attains its minimum at all points of an edge (i, k). Then each of the two components $\tilde{C}_i$, $\tilde{C}_k$ of the graph obtained from T by deleting the edge (i, k) has $n/2$ vertices (thus, n is even).*

Let us now complete the proof the Theorem 2.1.

Define a point M_0 in T_m as the point w in the case that (i) of Corollary 2.3 occurs and as the midpoint of the edge (i, k) in the case that (ii) occurs.

By minimality,

$$\sum(T)\leqslant S(M_0).$$

Suppose that for M_0 case (i) occurs. Define a function $F(t)$ for positive integers t by

$$F(t)=1+4+9+\cdots+t^2$$
$$(=\tfrac{1}{6}t(t+1)(2t+1)).$$

If $C_k=(V_k, E_k)$, $k=1,\ldots t$, are the components from (i), define $s_k=|V_k|$ and suppose

$$s_1\leqslant s_2\leqslant\cdots\leqslant s_t. \tag{11}$$

By Corollary 2.3,

$$s_t\leqslant\frac{n-1}{2}, \tag{12}$$

and also

$$\sum_{k=1}^{t} s_k=n-1.$$

In addition,

$$\sum_{x\in V_k} d(M_0, x)\leqslant F(s_k) \tag{13}$$

since for the vertices of the longest path $M_0=(u_1, u_2, \ldots, u_r)$ from M_0 to a vertex in V_k, the squares of the first r distances $1+4+\cdots+r^2$ are attained, whereas the squares of the distances of the remaining s_k-r vertices in V_k from M_0 are majorized successively by $(r+1)^2, \ldots, s_k^2$.

Note that for $t=2$

$$s_1=s_2=\frac{n-1}{2},$$

so that n is odd.

Let us now show that

$$\sum_{k=1}^{t} F(s_k)\leqslant 2F\left(\frac{n-1}{2}\right) \quad \text{if } n \text{ is odd,} \tag{14}$$

$$\sum_{k=1}^{t} F(s_k)\leqslant 2F\left(\frac{n}{2}-1\right)+1 \quad \text{if } n \text{ is even.} \tag{15}$$

Indeed, it is easily checked that if t_1, t_2 are integers satisfying $1\leqslant t_1\leqslant t_2$, then

$$F(t_1)+F(t_2)<F(t_1-1)+F(t_2+1). \tag{16}$$

Let us perform the algorithm of successively changing the original sequence $s_1\leqslant s_2\leqslant\cdots\leqslant s_t$ into a new sequence by diminishing first its nonzero member by one and then increasing the last member smaller than $n/2$ (the whole $n/2$) by one. By (16), the sum

$$\sum_{k=1}^{t} F(s_k)$$

increases and the algoritm ends (not counting zeros) in $(n-1)/2$, $(n-1)/2$ if n is odd, and in 1, $(n/2)-1$, $(n/2)-1$ if n is even.

This proves (14) and (15); the right-hand side of (14) is, however, equal to $\frac{1}{12}n(n^2-1)$ and the right-hand side of (15) is less than this number, equal to the bound in (8). The proof in this case is thus complete.

Suppose finally that for M_0 case (ii) occurs. Then we obtain analogously for the two components $\tilde{C}_i$, $\tilde{C}_k$

$$S(M_0)\leqslant 2\left(\left(\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2+\cdots+\left(\frac{n}{2}\right)^2\right),$$

the right-hand side of which is, again, $\frac{1}{12}n(n^2-1)$.

We are now able to prove the following theorem.

Theorem 2.4. *For any tree T with n vertices, $\hat{a}(T)$ is a rational number of the form*

$$\frac{n(n-1)}{C},$$

where C is an integer satisfying

$$n(n-1) \leqslant C \leqslant \tfrac{1}{12}n^2(n^2-1). \tag{17}$$

Both bounds are sharp, attained for the path P_n on the right-hand side and for the star S_n on the left-hand side. In addition, C is even unless both classes of bipartiteness of T have an odd number of vertices.

If the absolute centre of gravity M of T is in a vertex, $\hat{a}(T)$ has the form

$$\frac{n-1}{K},$$

where K is an integer satisfying

$$n-1 \leqslant K \leqslant \tfrac{1}{12}n(n^2-1).$$

In addition, if n is odd, K is even if and only if the class of bipartiteness containing M is odd.

Proof. We shall use formula (5) and distinguish two cases:

(a) The absolute centre of gravity M is in a vertex. Then $S(M)$ is an integer, and it suffices to show that if n is odd then $S(M)$ is even if and only if the class of bipartiteness containing M is odd. This is, however, an easy consequence of the fact that $S(M) = \sum_{x \in V} d(M, x) \bmod 2$, and this sum is even if and only if the class not containing M is even.

(b) The point M is an interior point of an edge (i, k); denote by V_1, V_2 the subsets of vertices of the components after deleting (i, k) from T, $i \in V_1$, $k \in V_2$. We set

$$\Sigma_i = \sum_{x \in V_1} d(i, x), \quad \Sigma_k = \sum_{x \in V_2} d(k, x).$$

Since

$$S(i) - S(k) = 2\Sigma_i - 2\Sigma_k + |V_2| - |V_1|,$$

$$|V_1| + |V_2| = n,$$

we conclude that $S(i) - S(k) \equiv n \bmod 2$, so that, by (7), $S(M)$ has the form C/n, C integer. The bounds for C follow from Theorem 2.1. It remains to investigate the parity of C. If n is odd, the number $\frac{1}{2}(n + S(k) - S(i))$ being an integer,

$$C = n\,S(k) - |\tfrac{1}{2}(n + S(k) - S(i))|^2$$

is congruent mod 2 to

$$S(k) - \tfrac{1}{2}(n + S(k) - S(i)),$$

which is easily seen to be even.

If n is even, the number

$$\begin{aligned} C &\equiv \tfrac{1}{2}(n+S(k)-S(i)) \\ &\equiv \tfrac{1}{2}(n-2\Sigma_i+2\Sigma_k-|V_2|+|V_1|) \\ &\equiv |V_1|+\Sigma_i-\Sigma_k. \end{aligned}$$

Let now B_1, B_2 be the classes of bipartiteness containing i, k, respectively. Then, mod 2,

$$\Sigma_i \equiv |V_1 \cap B_1|,$$

$$\Sigma_k \equiv |V_2 \cap B_2|,$$

so that

$$\begin{aligned} C &\equiv |V_1|+|V_1 \cap B_1|-|V_2 \cap B_2| \\ &\equiv |V_1 \cap B_2|+|V_2 \cap B_2| \\ &\equiv |B_2| \text{ (thus, } \equiv |B_1|). \end{aligned}$$

It follows that C is odd if and only if both classes of bipartiteness are odd, as asserted.

3. Absolute diameter and absolute radius of a graph

For an edge-valuated graph $G_C=(V,E,C)$, the distance of two vertices i, k is the number

$$d_c(i,k)=\min_{j_1,\dots,j_s}(c_{ij_1}+c_{j_1j_2}+\cdots+c_{j_{s-1}j_s}+c_{i_sk}),$$

where the minimum is taken for all possible paths from i to k in G.

The diameter of a connected graph G_C being

$$D(G_C)=\max_{i,k\in V} d_C(i,k),$$

we define the absolute diameter of G as

$$\hat{D}(G)=\min_{C\in\mathscr{C}(G)} D(G_C). \tag{18}$$

Similarly, the radius of G_C is

$$r(G_C)=\min_{x\in V}\max_{k\in V} d_C(x,k).$$

The absolute radius $\hat{r}(G)$ is then defined as

$$\hat{r}(G)=\min_{C\in\mathscr{C}(G)} r(G_C).$$

We shall prove the following theorem.

Theorem 3.1. *Let $G=(V, E)$ be a connected graph with n vertices. If G contains a circuit, then*

$$\hat{D}(G)=0. \tag{19}$$

If G does not contain a circuit (hence, is a tree) and has s end vertices, then

$$\hat{D}(G)=\frac{2(n-1)}{s}. \tag{20}$$

Proof. If G contains a circuit, let (i, k) be one of the edges of the circuit. The valuation assigning the value $|E|$ to the edge (i, k) and zero to all other edges leads to minimal $D(G_C)$, namely, to zero.

Let now $G=(V, E)$ be a tree with s endvertices $x_1, \ldots, x_s$. Let $C\in e(G)$, $C=(c_{ij})$. Clearly,

$$\sum_{\substack{i,j=1\\ i<j}}^{s} d(x_i, x_j)\leqslant\binom{s}{2}D(G_C). \tag{21}$$

On the other hand, every edge (p, q) is contained in $t(s-t)$ paths between endvertices if there are t endvertices in the branch containing p.

Since $t(s-t)\geqslant s-1$, we obtain

$$(s-1)\sum_{(p,q)\in E} c_{pq}\leqslant\sum_{\substack{i,j\\ i<j}} d(x_i, x_j),$$

which, together with (21), yields

$$(s-1)(n-1)\leqslant\binom{s}{2}D(G_C),$$

i.e.

$$D(G_C)\geqslant\frac{2(n-1)}{s}. \tag{22}$$

However, the edge valuation which assigns to each pending edge the value $(n-1)/s$ and to every other edge zero has the property that equality is attained in (22). This proves (20). □

Since for a tree G, $r(G_C)$ is always equal to $\frac{1}{2}D(G_C)$, we obtain that

$$\hat{r}(G)=0$$

for every connected graph containing a circuit, whereas

$$\hat{r}(G)=\frac{n-1}{s}$$

if G is a tree with s endvertices.

Remark. Since formula (19) does not have a good meaning, it would be more appropriate to consider in this case only such edge valuations $\mathscr{C}_m$ which satisfy the polygonal inequality: the value on an edge in a circuit is less than or equal to the sum of the values on the remaining edges in the circuit.

In such a case, (20) would still be true for the tree. However, (19) would no more be true. We can leave it as a problem to find the optimal value

$$\hat{D}_m(G) = \min_{C \in \mathscr{C}_m(G)} D(G_C)$$

for a general graph.

References

[1] M. Fiedler, Algebraische Zusammenhangszahl und ihre numerische Bedeutung, ISNM 26 (Birkhäuser, Basel, 1975) 69–85.

[2] M. Fiedler, Laplacian of graphs and algebraic connectivity, in: Combinatorics and Graph Theory, Banach Center Publ. Vol. 25 (PWN, Warszaw, 1989) 57–70.

[3] M. Fiedler, Absolute algebraic connectivity of trees, Linear and Multilinear Algebra 26 (1990) 85–106.

[4] M. Fiedler, A minimax problem for graphs and its relation to generalized doubly stochastic matrices. Linear and Multilinear Algebra 27 (1990) 1–23.

[5] M. Fiedler, An extremal problem for the spectral radius of a graph, Discrete Math. 108 (1992) 149–158.

[6] S.L. Hakimi, Optimum location of switching centers and the absolute center and medians of a graph, Oper. Res. 12 (1964) 450–459.

[7] S. Mohar and S. Poljak, Eigenvalues and the max-cut problem, Czechoslovak Math. J. 40 (115) (1990) 343–352.